HOMOCLINIC ORBITS OF FIRST-ORDER SUPERQUADRATIC HAMILTONIAN SYSTEMS

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ABSTRACT. In this article, we study the existence of homoclinic orbits for the first-order Hamiltonian system

$$J\dot{u}(t) + \nabla H(t, u(t)) = 0, \quad t \in \mathbb{R}.$$

Under the Ambrosetti-Rabinowitz's superquadraticy condition, or no Ambrosetti-Rabinowitz's superquadracity condition, we present two results on the existence of infinitely many large energy homoclinic orbits when H is even in u. We apply the generalized (variant) fountain theorems due to the author and Colin. Under no Ambrosetti-Rabinowitz's superquadracity condition, we also obtain the existence of a ground state homoclinic orbit by using the method of the generalized Nehari manifold for strongly indefinite functionals developed by Szulkin and Weth.

1. Introduction

In this paper, we are interrested in the existence and also the multiplicity of homoclinic orbits of the first order Hamiltonian system

$$J\dot{u}(t) + \nabla H(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{HS}$$

where $u \in \mathbb{R}^{2N}$, $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard symplectic structure on $\mathbb{R}^{2N}$, $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is $1-$periodic with respect to the $t-$variable and ∇H is the gradient of H with respect to u. We consider the case which H has the form

$$H(t, u) = \frac{1}{2}A(t)u \cdot u + W(t, u),$$

where the **dot** denotes the inner product of $\mathbb{R}^{2N}$, $A : \mathbb{R} \to \mathbb{R}^{4N^2}$ is a $2N \times 2N$ symmetric matrix-valued function and $W : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ satisfy the following conditions.

(A_0) $A \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^{4N^2})$ is $1-$periodic with respect to t and 0 lies in a gap of the spectrum $\sigma(L)$ of $L := -J\frac{d}{dt} - A(t)$.

(W_1) $W \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is $1-$periodic with respect to t and $W(t, 0) = 0$ for every $t \in \mathbb{R}$.

(W_2) There exist $c > 0$, $2 < p < \infty$ such that $|\nabla W(t, u)| \leqslant c\big(1 + b(t)|u|^{p-1}\big)$, where $b > 0$, $b \in L^\infty(\mathbb{R}) \cap L^r(\mathbb{R})$, $\frac{1}{r} + \frac{p}{s} = 1$ with $2 < s < \infty$.

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(W_3) $|\nabla W(t,u)| = \circ(|u|)$ as $|u| \to 0$ uniformly in t.

By homoclinic orbit of (HS) we mean a solution u satisfying

$$u \neq 0 \quad \text{and} \quad u(t) \to 0 \text{ as } t \to \infty.$$

In recent years, the existence and multiplicity of homoclinic orbits for the first order system (HS) were studied extensively by means of critical point theory, see for instance [6, 11, 13, 9, 24, 12, 8, 18, 7, 19, 25, 26]. In their seminal paper [7], Coti Zelati, Ekeland and Séré first considered (HS) with A a constant matrix, 0 is not a spectrum point of the Hamiltonian operator $L = -J\frac{d}{dt} - A(t)$, and $W(t,u)$ is convex in u and satisfies the Ambrosetti-Rabinowitz's condition

$$\exists \mu > 2, \quad 0 < \mu W(t,u) \leqslant u \cdot \nabla W(t,u), \quad \forall u \neq 0, \tag{AR}$$

which is extensively used in the study of superquadratic Hamiltonian systems. They proved the existence of two geometrically distinct homoclinic orbits for (HS). Subsequently, Séré [19] obtained the existence of infinitely homoclinic orbits for (HS) under more general assumption on W. In [25], Tanaka removed the convexity assumption and obtained the existence of at least one homoclinic orbit by using a subharmonic approach.

In this paper we first show, under condition (AR), that (HS) has infinitely many homoclinic orbits. We apply the generalized fountain theorem for strongly indefinite functionals established by the author and Colin [4]. More precisely, we have the following result.

Theorem 1. *Assume that* (A_0), $(W_1) - (W_4)$ *are satisfied. If in addition*

(W_4) $W(t,-u) = W(t,u)$,

(W_5) $\exists \mu > \max\{2, p-1\}$, $\exists \delta > 0$; $\delta |u|^\mu \leqslant \mu W(t,u) \leqslant u \cdot \nabla W(t,u)$,

then (HS) *has infinitely many large energy homoclinic solutions.*

Remark 2. *The existence of infinitely many homoclinic orbits of* (HS) *under* (A_0) *and* $(W_1) - (W_4)$ *was first proved by Ding and Girardi* [9] (*see also* [2]) *by using a generalized linking theorem. They allowed* 0 *to be an end point of the spectrum* $\sigma(L)$. *However, they assumed in addition that*

$$\exists c_0, \varepsilon_0 > 0 \ ; \ |\nabla W(t,u+v) - \nabla W(t,u)| \leqslant c_0 |v|(1 + |u|^{p-1}), \ \text{ whenever } |v| \leqslant \varepsilon_0.$$

Moreover, we do not know if the homoclinic orbits they obtained are large energy solutions of (HS).

It is well known that condition (AR) is mainly used, in superquadratic problems, to assure the boundedness of the Palais-Smale sequences of the energy functional, and without it the problem becomes more complicated. By applying a generalized linking theorem in the spirit of Krysewski and Szulkin [14], Wang et al. [26] obtained, without condition (AR), the existence of at least one homoclinic orbit of (HS). Subsequently, by replacing condition (AR) with a general superquadratic condition, Chen and Ma [6] obtained the existence of at least one homoclinic orbit of (HS) which is a ground state solution, that is a non zero solution which least energy. They adapted an earlier argument used by Yang [28] in the study of ground state solutions for a semilinear Schrödinger equation with periodic potential. By replacing in this paper (AR) with a general superquadratic condition, we also prove the following results.

Theorem 3. *Assume that* (A_0), $(W_1)-(W_4)$ *are satisfied. Assume in addition that* W *satisfies the following conditions.*

(W_6) $\big(v\cdot\nabla W(t,u)\big)(u\cdot v)\geqslant 0$.

(W_7) $\exists\gamma>2$ *such that* $\frac{W(t,u)}{|u|^\gamma}\to\infty$ *as* $|u|\to\infty$, *uniformly in* t.

(W_8) $W(t,u)>0$ *and* $u\cdot\nabla W(t,u)>2W(t,u)$, $\forall u\neq 0$.

(W_9) *if* $|u|=|v|$, *then* $W(t,u)=W(t,v)$ *and* $v\cdot\nabla W(t,u)\leqslant u\cdot\nabla W(t,u)$, *with strict inequality if* $u\neq v$.

(W_{10}) $|u|\neq|v|$ *and* $u\cdot v\neq 0\Rightarrow v\cdot\nabla W(t,u)\neq u\cdot\nabla W(t,v)$.

Then (HS) *has infinitely many large energy homoclinic solutions.*

Theorem 4. *If* (A_0), $(W_1)-(W_3)$, $(W_6)-(W_{10})$ *are satisfied, then* (HS) *has a homoclinic orbit which is a ground state solution.*

Remark 5. *In Theorem 4, the assumptions* (W_2) *and* (W_7) *can be weaken by taking* $b\equiv 1$ *and* $\gamma=2$ *respectively.*

As far as we know Theorem 3 is new. It will be proved by using the generalized variant fountain theorem due to author and Colin [3], which combines the $\tau-$topology of Kryszewski and Szulkin [14] with the idea of the monotonicity trick for strongly indefinite functionals inspired by Jeanjean [15]. Theorem 4 was first proved by Chen and Ma [6]. They applied a generalized weak linking theorem due to Schechter and Zou [20]. In this paper we follow a different approach, which is based on the method of the generalized Nehari manifold for strongly indefinite functionals inspired by Pankov [17], and developed recently by Szulkin and Weth [22, 23]. This approach is much more direct and simpler.

The paper is organized as follows. The variational framework for the study of (HS) will be stated in section 2, while the existence of infinitely many large energy homoclinic orbits will be proved in Section 3. Finally, in Section 4 we apply the method of the generalized Nehari manifold to find a ground state homoclinic orbit of (HS).

2. Variational setting

Let $X:=H^{\frac{1}{2}}(\mathbb{R},\mathbb{R}^{2N})$ be the fractional Sobolev space of functions $u\in L^2(\mathbb{R},\mathbb{R}^{2N})$ such that

$$\int_{\mathbb{R}}(1+\xi^2)|\mathcal{F}u(\xi)|^2d\xi<\infty,$$

where $\mathcal{F}$ is the Fourier transform. X is a separable Hilbert space with the inner product

$$\big\langle u,v\big\rangle_{\frac{1}{2}}:=\int_{\mathbb{R}}(1+\xi^2)^{\frac{1}{2}}\mathcal{F}u(\xi)\overline{\mathcal{F}v(\xi)}d\xi,\quad u,v\in X.$$

For $q\in[2,+\infty[$, the Sobolev embedding $X\hookrightarrow L^q(\mathbb{R},\mathbb{R}^{2N})$ is continuous and the embedding $X\hookrightarrow L^q_{loc}(\mathbb{R},\mathbb{R}^{2N})$ is compact $\big($see for example [1] or [21]$\big)$.
Consider the operator $B:X\to X$ defined by

$$\big\langle Bu,v\big\rangle_{\frac{1}{2}}:=\int_{\mathbb{R}}\big(-J\dot{u}-A(t)u\big)\cdot v dt.$$

By assumption (A_0), $L = -J\frac{d}{dt} - A(t) : L^2(\mathbb{R},\mathbb{R}^{2N}) \to L^2(\mathbb{R},\mathbb{R}^{2N})$ is a selfadjoint bounded operator with domain $D(L) = H^1(\mathbb{R},\mathbb{R}^{2N})$, and the spectrum is unbounded below and above in $H^1(\mathbb{R},\mathbb{R}^{2N})$ $\big($see [21]$\big)$. Hence, the space X has the orthogonal decomposition $X = X^+ \oplus X^-$, where $X^\pm$ are infinite dimensional $B-$invariant subspaces such that the quadratic form $u \in X \mapsto \langle Bu, u\rangle$ is negative on X^- and positive on X^+. Therefore we can define a new equivalent inner product on X by setting

$$\langle u, v\rangle := \langle Bu^+, v^+\rangle_{\frac{1}{2}} - \langle Bu^-, v^-\rangle_{\frac{1}{2}}, \quad u^\pm, v^\pm \in X^\pm.$$

If $\|\cdot\|$ denotes the corresponding norm, then we have

$$\int_{\mathbb{R}} \big(- J\dot{u} - A(t)u\big) \cdot u dt = \|u^+\|^2 - \|u^-\|^2, \quad \forall u \in X.$$

We define on X the functional

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}} \big(- J\dot{u} - A(t)u\big) \cdot u dt - \int_{\mathbb{R}} W(t,u)dt.$$

Then

$$\Phi(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}} W(t,u)dt. \tag{1}$$

Proposition 6 ([2], Proposition 3.1)**.** *If (W_1), (W_2) and (W_3) are satisfied, then $\Phi \in \mathcal{C}^1(X,\mathbb{R})$, with*

$$\langle \Phi'(u), v\rangle = \langle u^+, v\rangle - \langle u^-, v\rangle - \int_{\mathbb{R}} v \cdot \nabla W(t,u)dt.$$

Moreover, $u \in X$ is a homoclinic orbit of (HS) *if and only if it is a non zero critical point of Φ.*

Due to the periodicity of A and W, if $u = u(t)$ is a homoclinic orbit of (HS), so are all $g \star u$, $g \in \mathbb{Z}$, where

$$g \star u(t) := u(t-g).$$

Therefore the functional Φ cannot satisfy the Palais-Smale condition at any critical level $c \neq 0$. We recall that a functional $\varphi \in \mathcal{C}^1(X,\mathbb{R})$ is said to satisfy the Palais-Smale condition $\big($resp. the Palais-Smale condition at level $c \in \mathbb{R}\big)$, if every sequence $(u_n) \subset X$ such that $(\varphi(u_n))_n$ is bounded $\big($resp. $\varphi(u_n) \to c\big)$ and $\varphi'(u_n) \to 0$, admits a convergent subsequence. Two homoclinic orbits u and v of (HS) are said to be geometrically distinct if the sets $\{g \star u\,;\, g \in \mathbb{Z}\}$ and $\{g \star v\,;\, g \in \mathbb{Z}\}$ are disjoint.

3. The existence of infinitely many homoclinic orbits

3.1. **Generalized (variant) fountain theorems.** Let Y be a closed subspace of a separable Hilbert space X endowed with the inner product $(\cdot)$ and the associated norm $\|\cdot\|$. We denote by $P : X \to Y$ and $Q : X \to Z := Y^\perp$ the orthogonal projections.

We fix an orthonormal basis $(a_j)_{j\geqslant 0}$ of Y and we consider on $X = Y \oplus Z$ the $\tau-$topology introduced by Kryszewski and Szulkin in [14]; that is the topology associated to the following norm

$$|||u||| := \max\Big(\sum_{j=0}^{\infty} \frac{1}{2^{j+1}}|(Pu, a_j)|, \|Qu\|\Big),\ u \in X.$$

τ has the following interesting property (see [14] or [27]): If $(u_n) \subset X$ is a bounded sequence, then

$$u_n \xrightarrow{\tau} u \Longleftrightarrow Pu_n \rightharpoonup Pu \ \text{and}\ Qu_n \to Qu.$$

Let $(e_j)_{j\geqslant 0}$ be an orthonormal basis of Z. We adopt the following notations:

$$Y_k := Y \oplus (\oplus_{j=0}^{k}\mathbb{R}e_j) \qquad \text{and} \qquad Z_k := \overline{\oplus_{j=k}^{\infty}\mathbb{R}e_j}.$$

$B_k := \{u \in Y_k \,\big|\, ||u|| \leqslant \rho_k\}$, $\quad N_k := \{u \in Z_k \,\big|\, ||u|| = r_k\}$ *where* $\ 0 < r_k < \rho_k,\ k \geqslant 2.$
The following abstract critical point theorems are due to the author and F. Colin.

Theorem 7 (Fountain theorem, Batkam-Colin [4])**.** *Let $\Phi \in \mathcal{C}^1(X,\mathbb{R})$ be an even functional which is τ-upper semicontinuous and such that Φ' is weakly sequentially continuous. If there exist $\rho_k > r_k > 0$ such that:*

(A_1) $\ a_k := \sup\limits_{\substack{u\in Y_k\\ \|u\|=\rho_k}} \Phi(u) \leqslant 0 \quad \text{and} \quad \sup\limits_{\substack{u\in Y_k\\ \|u\|\leqslant\rho_k}} \Phi(u) < \infty.$

(A_2) $\ b_k := \inf\limits_{\substack{u\in Z_k\\ \|u\|=r_k}} \Phi(u) \to \infty,\ k\to\infty.$

Then

$$c_k := \inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} \Phi(\gamma(u)) \ \geqslant\ b_k,$$

and there exists a sequence $(u_k^n)_n \subset X$ such that

$$\Phi'(u_k^n) \to 0 \quad \text{and} \quad \Phi(u_k^n) \to c_k \quad \text{as}\ \ n\to\infty,$$

where Γ_k is the set of maps $\gamma : B_k \to X$ such that

(a) *γ is odd and $\tau-$continuous, and $\gamma_{|_{\partial B_k}} = id$,*
(b) *every $u \in int(B_k)$ has a $\tau-$neighborhood N_u in Y_k such that $(id-\gamma)(N_u \cap int(B_k))$ is contained in a finite-dimensional subspace of X,*
(c) *$\Phi(\gamma(u)) \leqslant \Phi(u)\ \forall u \in B_k$.*

Theorem 8 (Variant fountain theorem, Batkam-Colin [3])**.** *Let the family of $\mathcal{C}^1$-functionals*

$$\Phi_\lambda : X \to \mathbb{R}, \quad \Phi_\lambda(u) := L(u) - \lambda J(u), \qquad \lambda \in [1,2],$$

such that

(B_1) *Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for every $(\lambda,u) \in [1,2]\times X$.*
(B_2) *$J(u) \geqslant 0$ for every $u \in X$; $L(u) \to \infty$ or $J(u) \to \infty$ as $\|u\| \to \infty$.*
(B_3) *For every $\lambda \in [1,2]$, Φ_λ is τ-upper semicontinuous and Φ'_λ is weakly sequentially continuous.*

If there are $0 < r_k < \rho_k$ such that

$$b_k(\lambda) := \inf_{\substack{u\in Z_k\\ \|u\|=r_k}} \Phi_\lambda(u) \ \geqslant\ a_k(\lambda) := \sup_{\substack{u\in Y_k\\ \|u\|=\rho_k}} \Phi_\lambda(u)\ \forall\lambda\in[1,2],$$

then

$$c_k(\lambda) := \inf_{\theta\in\Theta_k(\lambda)}\sup_{u\in B_k} \Phi_\lambda\big(\theta(u)\big) \ \geqslant\ b_k(\lambda)\ \ \forall\lambda\in[1,2].$$

Moreover, for a.e $\lambda \in [1,2]$ there exists a sequence $(u_k^n(\lambda))_n \subset X$ such that

$$\sup_n \|u_k^n(\lambda)\| < \infty, \quad \Phi'_\lambda(u_k^n(\lambda)) \to 0 \ \text{and}\ \Phi_\lambda(u_k^n(\lambda)) \to c_k(\lambda) \text{ as } n\to\infty.$$

Where $\Theta_k(\lambda)$ is the class of maps $\theta : B_k \to X$ such that

(a) *θ is odd and $\tau-$continuous, and $\theta_{|\partial B_k} = id$,*
(b) *every $u \in int(B_k)$ has a $\tau-$neighborhood N_u in Y_k such that $(id - \theta)(N_u \cap int(B_k))$ is contained in a finite-dimensional subspace of X,*
(c) *$\Phi_\lambda(\theta(u)) \leqslant \Phi_\lambda(u)$ $\forall u \in B_k$.*

In the following two subsections, we set $Y = X^+$ and $Z = X^-$.

3.2. **The case of Ambrosetti-Rabinowitz condition.** In this subsection we, assume that (A_0), $(W_1) - (W_5)$ are satisfied.
The functional Φ reads as follows:

$$\Phi(u) = \frac{1}{2}\|Qu\|^2 - \frac{1}{2}\|Pu\|^2 - \int_{\mathbb{R}} W(t,u)dt. \tag{2}$$

We know from Proposition 6 that Φ is of class $\mathcal{C}^1$ on X and

$$\langle \Phi'(u), v\rangle = \langle Qu, v\rangle - \langle Pu, v\rangle - \int_{\mathbb{R}} v \cdot \nabla W(t,u)dt. \tag{3}$$

We have the following lemma.

Lemma 9. *Φ is $\tau-$upper semicontinuous on X, and Φ' is weakly sequentially continuous.*

Proof. Let $u_n \xrightarrow{\tau} u$ in X and $\Phi(u_n) \geqslant C \in \mathbb{R}$. Then, by the definition of τ we have $Qu_n \to Qu$, and then (Qu_n) is bounded. Since $W \geqslant 0$, we deduce from the inequality $C \leqslant \Phi(u_n)$ that (Pu_n) is also bounded. hence, $u_n \rightharpoonup u$ in X, $u_n \to u$ in $L^p_{loc}(\mathbb{R}, \mathbb{R}^{2N})$, and up to a subsequence $u_n(t) \to u(t)$ a.e $t \in \mathbb{R}$. It follows from Fatou's lemma and the weakly semicontinuity of the norm $\|\cdot\|$ that $C \leqslant \Phi(u)$. Hence Φ is $\tau-$upper semicontinuous.

Now assume that $u_n \rightharpoonup u$ in X. Then $u_n \to u$ in $L^p_{loc}(\mathbb{R}, \mathbb{R}^{2N})$, and since $b(t) \leqslant \|b\|_\infty$ a.e. t, we deduce from Theorem $A.2$ of [27] that $\nabla W(t,u_n) \to \nabla W(t,u)$ in $L^{p/(p-1)}_{loc}(\mathbb{R}, \mathbb{R}^{2N})$. hence $\langle \Phi'(u_n), v\rangle \to \langle \Phi'(u), v\rangle$ for all $v \in C^\infty_c(\mathbb{R}, \mathbb{R}^{2N})$. We then deduce by density that Φ' is weakly sequentially continuous. □

Lemma 10. *Every Palais-Smale sequence for Φ is bounded.*

Proof. Let $(u_n) \subset X$ and $d \in \mathbb{R}$ such that $\sup|\Phi(u_n)| \leqslant d$ and $\Phi'(u_n) \to 0$.
It follows from (W_5) that

$$\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle \geqslant (\frac{\mu}{2} - 1)\delta|u|^\mu_\mu.$$

We deduce that for n big enough

$$(\frac{\mu}{2} - 1)\delta|u|^\mu_\mu \leqslant d + \|u_n\|. \tag{4}$$

On the other hand (W_2) and (W_3) imply that

$$\forall \varepsilon > 0,\ \exists c(\varepsilon) > 0;\ \ |\nabla W(t,u)| \leqslant \varepsilon|u| + c(\varepsilon)|u|^{p-1}\ a.e.\, t \in \mathbb{R},\ \forall u \in \mathbb{R}^{2N}. \tag{5}$$

Hence

$$\begin{aligned}\|Qu_n\|^2 &= \langle \Phi'(u_n), Qu_n\rangle + \int_{\mathbb{R}} Qu_n \cdot W(t,u_n)dt \\ &\leqslant \|Qu_n\| + \int_{\mathbb{R}} Qu_n \cdot W(t,u_n)dt \quad \text{(for n big enough)} \\ &\leqslant \|Qu_n\| + \varepsilon \int_{\mathbb{R}} |Qu_n||u_n|dt + c(\varepsilon)\int_{\mathbb{R}} |Qu_n||u_n|^{p-1}dt.\end{aligned}$$

By the same way we have

$$\|Pu_n\|^2 \leqslant \|Pu_n\| + \varepsilon \int_{\mathbb{R}} |Pu_n||u_n|dt + c(\varepsilon)\int_{\mathbb{R}} |Pu_n||u_n|^{p-1}dt.$$

And by using the Höder inequality and the Sobolev embedding theorem we obtain

$$\|Qu_n\|^2 + \|Pu_n\|^2 \leqslant \|Qu_n\| + \|Pu_n\| + c_1\varepsilon\|u_n\|^2 + c_2 c(\varepsilon)\|u_n\||u_n|_\mu^{p-1}.$$

By taking (4) into account we get

$$\|Qu_n\|^2 + \|Pu_n\|^2 \leqslant \|Qu_n\| + \|Pu_n\| + c_1\varepsilon\|u_n\|^2 + c_3 c(\varepsilon)\|u_n\|\big(1 + \|u_n\|^{\frac{p-1}{\mu}}\big).$$

Hence,

$$(1 - c_1\varepsilon)\|u_n\|^2 \leqslant \|Qu_n\| + \|Pu_n\| + c_3 c(\varepsilon)\|u_n\|\big(1 + \|u_n\|^{\frac{p-1}{\mu}}\big).$$

Since by (W_5) we have $\frac{p-1}{\mu} < 1$, it then suffices to fix $\varepsilon < \frac{1}{2c_1}$ to conclude. □

The following lemma will be helpful for our arguments. It is a special case of a more general result due to P. L. Lions [16].

Lemma 11. *Let (u_n) be a bounded sequence in X. If there is $r > 0$ such that*

$$\lim_{n\to\infty} \sup_{a\in\mathbb{R}} \int_{-a}^{+a} |u_n|^2 = 0,$$

then $u_n \to 0$ in $L^q(\mathbb{R}, \mathbb{R}^{2N})$ for all $q \in (2, \infty)$.

Proof of Theorem 1. Let $u \in Y_k$. (W_5) implies that

$$\Phi(u) \leqslant \frac{1}{2}\|Qu\|^2 - \frac{1}{2}\|Pu\|^2 - c|u|_\mu^\mu.$$

Let $\widehat{Y_k}$ be the closure of Y_k in $L^\mu(\mathbb{R}, \mathbb{R}^{2N})$, then there is a continuous projection of $\widehat{Y_k}$ on $\oplus_{j=0}^k \mathbb{R}e_j$, and since all norms are equivalent on the latter space we can find a constant $c_1 > 0$ such that $c_1\|Qu\|^\mu \leqslant |u|_\mu^\mu$. It follows that

$$\Phi(u) \leqslant \frac{1}{2}\|Qu\|^2 - \frac{1}{2}\|Pu\|^2 - c_1\|Qu\|^\mu.$$

This implies that $\Phi(u) \to -\infty$ as $\|u\| \to \infty$, and condition (A_1) of Theorem 7 is therefore satisfied for ρ_k large enough.

Now let $u \in Z_k$. We deduce from (W_2) and (W_3) that

$$\forall\varepsilon > 0,\ \exists c_\varepsilon > 0\,;\ |\nabla W(t,u)| \leqslant \varepsilon|u| + c_\varepsilon b(t)|u|^{p-1}. \tag{6}$$

It then follows that

$$\Phi(u) \geqslant \frac{1}{2}(1 - c\varepsilon)\|u\|^2 - c_\varepsilon \int_{\mathbb{R}} \frac{b(t)}{p}|u|^p dt.$$

By choosing $\varepsilon = \frac{1}{2c}$ we obtain

$$\Phi(u) \geqslant \frac{1}{4}\|u\|^2 - c_1 \int_{\mathbb{R}} \frac{b(t)}{p}|u|^p dt.$$

Let

$$\beta_k := \sup_{\substack{u\in Z_k\\ \|u\|=1}} \Big(\int_{\mathbb{R}} \frac{b(t)}{p}|u|^p dt\Big)^{\frac{1}{p}}. \tag{7}$$

Then

$$\Phi(u) \geqslant \frac{1}{4}\|u\|^2 - c_1\beta_k^p\|u\|^p = \frac{1}{2}\Big(\frac{1}{2}\|u\|^2 - c_2\beta_k^p\|u\|^p\Big).$$

If we set $r_k := \big(c_2 p\beta_k\big)^{\frac{1}{2-p}}$, then for every $u \in Z_k$ such that $\|u\| = r_k$ we have

$$\Phi(u) \geqslant \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{p}\Big)\big(c_2 p\beta_k\big)^{\frac{2}{2-p}}.$$

By Lemma 12 below, $\beta_k \to 0$ as $k \to \infty$. Hence assumption (A_2) of Theorem 7 is satisfied.

By applying Theorem 7, we obtain the existence of a sequence $(u_k^n)_n \subset X$ such that $\Phi(u_k^n) \to c_k$ and $\Phi'(u_k^n) \to 0$ as $n \to \infty$, for every k.
We claim that there exist a sequence $(a_n) \subset \mathbb{R}$ and real numbers $r, \gamma > 0$ such that for k big enough

$$\liminf_{n\to\infty} \int_{-a_n}^{+a_n} |u_k^n|^2 \geqslant \gamma. \tag{8}$$

In fact, if the claim is not true then, because (u_k^n) is bounded by Lemma 10, we deduce from Lemma 11 the existence of a subsequence, still denoted (u_k^n), such that $u_k^n \to 0$ in $L^p(\mathbb{R}, \mathbb{R}^{2N})$. By using the Hölder inequality and (5) we have

$$\begin{aligned}\int_{\mathbb{R}} Pu_k^n \cdot \nabla W(t, u_k^n)dt &\leqslant \varepsilon|u_k^n|_2|Pu_k^n|_2 + c(\varepsilon)|u_k^n|_p^{p-1}|Pu_k^n|_p \\ &\leqslant C\big(\varepsilon + c(\varepsilon)|u_n|_p^{p-1}\big),\end{aligned}$$

where C is a constant which does not depend on n and ε. It follows that

$$\limsup_{n\to\infty} \int_{\mathbb{R}} Pu_k^n \cdot \nabla W(t, u_k^n)dt \leqslant c\varepsilon,$$

and since ε is arbitrary we deduce that

$$\int_{\mathbb{R}} Pu_k^n \cdot \nabla W(t, u_k^n)dt \to 0 \text{ as } n \to \infty.$$

By the same way we show that

$$\int_{\mathbb{R}} Qu_k^n \cdot \nabla W(t, u_k^n)dt \to 0 \text{ and } \int_{\mathbb{R}} W(t, u_k^n)dt \to 0 \text{ as } n \to \infty.$$

It then follows that

$$c_k = \lim_{n\to\infty} \big(\Phi(u_k^n) - \frac{1}{2}\big\langle\Phi'(u_k^n), u_k^n\big\rangle\big) = \lim_{n\to\infty} \int_{\mathbb{R}} \Big(\frac{1}{2}u_k^n \cdot \nabla W(t, u_k^n) - W(t, u_k^n)\Big)dt = 0.$$

We obtain a contradiction by taking k sufficiently large, since $c_k \geqslant b_k \to \infty$ as $k \to \infty$.
Now by (8) there exists a subsequence, still denoted (u_k^n), such that for k big enough

$$\|u_k^n\|_{L^2\big((\alpha_n - r, \alpha_n + r), \mathbb{R}^{2N}\big)} \geqslant \frac{\gamma}{2} \quad \forall n.$$

By a standard argument, there is $q_n \in \mathbb{Z}$ such that for k big enough

$$\|w_k^n\|_{L^2\left((-r-\frac{1}{2},r+\frac{1}{2}),\mathbb{R}^{2N}\right)} \geqslant \frac{\gamma}{2} \quad \forall n, \tag{9}$$

where $w_k^n := u_k^n(\cdot - q_n)$. Since both Φ and Φ' are invariant under translation, it follows that $\Phi(w_k^n) \to c_k$ and $\Phi'(w_k^n) \to 0$ as $n \to \infty$. By Lemma 10 again, the sequence (w_k^n) is bounded. Up to a subsequence, we may suppose that

$$w_k^n \rightharpoonup w_k \text{ in } X, \quad w_k^n \to w_k \text{ in } L^2_{loc}(\mathbb{R},\mathbb{R}^{2N}), \quad w_k^n \to w_k \text{ a.e.} \tag{10}$$

By (9), $w_k \neq 0$ for k large enough, and in view of the weak sequentially semicontinuity of Φ' we have $\Phi'(w_k) = 0$. That is, w_k is a critical point of Φ and therefore a weak solution of (HS). Again by (9), we have for $0 < R < \infty$ and k large enough

$$\sup_{a\in\mathbb{R}} \int_{a-R}^{a+R} |w_k^n - w_k|^2 \to 0, \ n \to \infty.$$

Lemma 11 then implies that $w_k^n \to w_k$ in $L^p(\mathbb{R},\mathbb{R}^{2N})$. Using this and (5), one can verify easily that

$$\int_{\mathbb{R}} (w_k^n - w_k)\cdot \nabla W(t, w_k^n - w_k)dt \to 0, \quad \int_{\mathbb{R}} W(t, w_k^n - w_k)dt \to 0, \quad n \to \infty.$$

By Brezis-Lieb lemma [5], we also have as $n \to \infty$

$$\int_{\mathbb{R}} w_k^n \cdot \nabla W(t, w_k^n)dt \to w_k \cdot \nabla W(t, w_k)dt, \quad \int_{\mathbb{R}} W(t, w_k^n)dt \to \int_{\mathbb{R}} W(t, w_k)dt.$$

By taking the limit $n \to \infty$ in the expression

$$\Phi(w_k^n) = \left\langle \Phi'(w_k^n), w_k^n \right\rangle + \frac{1}{2}\int_{\mathbb{R}} w_k^n \cdot \nabla W(t, w_k^n)dt - \int_{\mathbb{R}} W(t, w_k^n)dt,$$

we therefore deduce that $\Phi(w_k) = c_k$. Since $c_k \geqslant b_k \to \infty$, $k \to \infty$, the theorem is proved. □

Lemma 12. *Assume that $b > 0$, $b \in L^\infty(\mathbb{R}) \cap L^r(\mathbb{R})$, $\frac{1}{r} + \frac{p}{s} = 1$ with $2 < p, s < \infty$. Then*

$$\beta_k = \sup_{\substack{u\in Z_k \\ \|u\|=1}} \Big(\int_{\mathbb{R}} \frac{b(t)}{p}|u|^p dt\Big)^{\frac{1}{p}} \to 0, \text{ as } k \to \infty.$$

Proof. Clearly $0 \leqslant \beta_{k+1} \leqslant \beta_k$, hence $\beta_k \to \beta \geqslant 0$. For every k, there exists $u_k \in Z_k$ such that

$$0 \leqslant \beta_k^p - \int_{\mathbb{R}} \frac{b(t)}{p}|u_k|^p dt < \frac{1}{k}.$$

Up to a subsequence we have $u_k \rightharpoonup u$ in X. By the definition of Z_k we have $u = 0$. Since X embeds continuously in $L^s(\mathbb{R},\mathbb{R}^{2N})$, the sequence (u_k) is also bounded in $L^s(\mathbb{R},\mathbb{R}^{2N})$. Therefore, there is a constant $C > 0$ such that $\||u_k|^p\|_{L^{s/p}(\mathbb{R},\mathbb{R}^{2N})} \leqslant C$. Let $I_k :=]-k,k[$.

$$b \in L^r(\mathbb{R},\mathbb{R}^{2N}) \Rightarrow \|b\|_{L^r(\mathbb{R}\setminus I_k)} \to 0, \quad \text{as } k \to \infty.$$

It follows that for every $\varepsilon > 0$, we can find k_1 large enough such that $\|b\|_{L^r(\mathbb{R}\setminus I_{k_1})} < \varepsilon$. Now since the embedding $H^1(I_{k_1},\mathbb{R}^{2N}) \hookrightarrow L^p(I_{k_1},\mathbb{R}^{2N})$ is compact, we have $u_k \to$

0 in $L^p(I_{k_1},\mathbb{R}^{2N})$, and since $b(t)\leqslant\|b\|_\infty$ a.e., we obtain in view of Theorem $A.2$ in [27] that $\int_{I_{k_1}}\frac{b(t)}{p}|u_k|^p dt\to 0$ as $k\to\infty$. So there is k_0 such that

$$\int_{I_k}\frac{b(t)}{p}|u_k|^p dt<\varepsilon,\quad \forall k\geqslant k_0.$$

Since $\frac{1}{p}+\frac{1}{s/p}=1$, we deduce from the Hölder inequality that

$$\begin{aligned}\int_{\mathbb{R}}\frac{b(t)}{p}|u_k|^p dt &= \int_{I_{k_1}}\frac{b(t)}{p}|u_k|^p dt+\int_{\mathbb{R}\setminus I_{k_1}}\frac{b(t)}{p}|u_k|^p dt\\ &\leqslant \int_{I_{k_1}}\frac{b(t)}{p}|u_k|^p dt+\frac{1}{p}\Big(\int_{\mathbb{R}\setminus I_{k_1}}(b(t))^r dt\Big)^r\Big(\int_{\mathbb{R}\setminus I_{k_1}}|u|^s dt\Big)^{p/s}\\ &= \int_{I_{k_1}}\frac{b(t)}{p}|u_k|^p dt+\frac{1}{p}\|b\|_{L^r(\mathbb{R}\setminus I_{k_1})}\||u|^p\|_{L^{s/p}(\mathbb{R}\setminus I_{k_1},\mathbb{R}^{2N})}\\ &< \varepsilon\big(1+C/p\big)\quad\big(\text{for } k \text{ big enough}\big).\end{aligned}$$

It follows that

$$\limsup_{k\to\infty}\int_{\mathbb{R}}\frac{b(t)}{p}|u_k|^p dt\leqslant\varepsilon\big(1+C/p\big).$$

We then conclude by taking the limit $\varepsilon\to 0$. □

3.3. **The case of a general superquadratic condition.** In this subsection, we assume that (A_0), $(W_1)-(W_4)$ and $(W_6)-(W_{10})$ are satisfied.
We define $\Phi_\lambda : X\to\mathbb{R}$ by

$$\Phi_\lambda(u)=\frac{1}{2}\|Qu\|^2-\lambda\Big[\frac{1}{2}\|Pu\|^2+\int_{\mathbb{R}}W(t,u)dt\Big],\quad \lambda\in[1,2]. \tag{11}$$

A standard argument shows that:

Lemma 13. *The conditions (B_1), (B_2) and (B_3) of Theorem 8 are satisfied, with*

$$L(u):=\frac{1}{2}\|Qu\|^2,\quad J(u):=\frac{1}{2}\|Pu\|^2+\int_{\mathbb{R}}W(t,u)dt.$$

Moreover, Φ'_λ is given by

$$\big\langle\Phi'_\lambda(u),v\big\rangle=\big\langle Qu,v\big\rangle-\lambda\Big[\big\langle Pu,v\big\rangle+\int_{\mathbb{R}}v\cdot\nabla W(t,u)dt\Big]. \tag{12}$$

Lemma 14. *For a.e. $\lambda\in[1,2]$, there exists $u_k(\lambda)\in X$ such that $\Phi_\lambda(u_k(\lambda))=c_k(\lambda)$ and $\Phi'(u_k(\lambda))=0$, for k big enough.*

Proof. The assumptions (W_3) and (W_7) imply that for every $\delta>0$, there is $C_\delta>0$ such that $W(t,u)\geqslant C_\delta|u|^\mu-\delta|u|^2$. It follows as in the proof of Theorem 1 that for every $u\in Y_k$, $\Phi_\lambda(u)\to-\infty$ as $\|u\|\to\infty$, uniformly in $\lambda\in[1,2]$. Therefore, we can choose ρ_k sufficiently large such that $a_k(\lambda)\leqslant 0$.

Let $u\in Z_k$. As in the proof of Theorem 1, we can show that for any $\lambda\in[1,2]$,

$$\Phi_\lambda(u)\geqslant\frac{1}{2}\Big(\frac{1}{2}-C\beta_k^p\|u\|^p\Big),$$

where $C>0$ is constant and β_k is defined by (7). If u is chosen such that $\|u\|=r_k:=\big(Cp\beta_k^p\big)^{\frac{1}{2-p}}$, then we have

$$\Phi_\lambda(u)\;\geqslant\;\widetilde{b_k}:=\frac{1}{2}\Big(\frac{1}{2}-\frac{1}{p}\Big)\big(Cp\beta_k^p\big)^{\frac{2}{2-p}}. \tag{13}$$

Since by Lemma 12 $\beta_k \to 0$, we have $\widetilde{b_k} \to \infty$ and hence $b_k(\lambda) \to \infty$ uniformly in λ, as $k \to \infty$.
By applying Theorem 8, we then conclude, for k large enough, that $c_k(\lambda) \geqslant b_k(\lambda)$ and for a.e. $\lambda \in [1,2]$, there exists a sequence $(v_k^n(\lambda))$ in X such that

$$\sup_n \|v_k^n\| < \infty, \quad \Phi_\lambda(v_k^n) \to c_k(\lambda), \quad \Phi_\lambda(v_k^n) \to 0, \text{ as } n \to \infty.$$

We proceed as in the proof of Theorem 1 to obtain the existence of $(u_k^n(\lambda))$ which satisfies the conclusion of the Lemma. ☐

As a consequence of the above lemma we have:

Corollary 15. *There exist* $(\lambda_n) \subset [1,2]$ *and* $(z_k^n)_n \subset X\backslash\{0\}$ *such that*

$$\lambda_n \to 1, \quad \Phi'_\lambda(z_k^n) = 0, \quad \Phi_\lambda(z_k^n) = c_k(\lambda_n).$$

We need the following lemma.

Lemma 16. *Let* $\lambda \in [1,2]$. *If* $z_\lambda \neq 0$ *and* $\Phi'_\lambda(z_\lambda) = 0$, *then* $\Phi_\lambda(z_\lambda + w) < \Phi_\lambda(z_\lambda)$ *for every* $w \in \mathcal{Z}_\lambda := \{rz_\lambda + v\,;\, r \geqslant -1, v \in Y\}$.

Proof. Let $w = rz_\lambda + v \in Z_\lambda$. It is easy to verify that

$$\Phi_\lambda(z_\lambda + w) - \Phi_\lambda(z_\lambda) = -\frac{\lambda}{2}\|v\|^2 + r(\frac{r}{2}+1)\|Qz_\lambda\|^2 - \lambda r(\frac{r}{2}+1)\|Pz_\lambda\|^2$$
$$- \lambda\Big[(1+r)\big\langle Pz_\lambda, v\big\rangle + \int_{\mathbb{R}} W\big(t,(1+r)z_\lambda + v\big)dt - \int_{\mathbb{R}} W(t,z_\lambda)\Big]. \quad (14)$$

Now, $\Phi'_\lambda(z_\lambda) = 0$ implies $\big\langle \Phi'_\lambda(z_\lambda), r(\frac{r}{2}+1)z_\lambda + (1+r)v\big\rangle = 0$, which gives

$$r(\frac{r}{2}+1)\|Qz_\lambda\|^2 - \lambda\Big[r(\frac{r}{2}+1)\|Pz_\lambda\|^2 + (1+r)\big\langle Pz_\lambda, v\big\rangle\Big] =$$
$$\lambda\int_{\mathbb{R}} \big(r(\frac{r}{2}+1)z_\lambda + (1+r)v\big)\cdot\nabla W(t,z_\lambda).$$

Reporting this in (14) we obtain

$$\Phi_\lambda(z_\lambda + w) - \Phi_\lambda(z_\lambda) = -\frac{\lambda}{2}\|v\|^2 +$$
$$\lambda\int_{\mathbb{R}} \Big[\big(r(\frac{r}{2}+1)z_\lambda + (1+r)v\big)\cdot\nabla W(t,z_\lambda) + W(t,z_\lambda) - W(t,z_\lambda + w)\Big]dt. \quad (15)$$

We define $f : [-1,\infty[\to \mathbb{R}$ by

$$f(s) := \big(s(\frac{s}{2}+1)z_\lambda + (1+s)v\big)\cdot\nabla W(t,z_\lambda) + W(t,z_\lambda) - W(t,z_\lambda + w).$$

Since $z_\lambda \neq 0$, then in view of (W_8) we have $f(-1) < 0$. On the other hand, we deduce from (W_7) and (W_8) that $f(s) \to -\infty$ as $s \to \infty$. Therefore, f attains its maximum at a point $s \in [-1,\infty[$ which satisfies

$$f'(s) = \big((1+s)z_\lambda + v\big)\cdot\nabla W(t,z_\lambda) - z_\lambda\cdot\nabla W(t,(1+s)z_\lambda + v) = 0. \quad (16)$$

Setting $y_\lambda = z_\lambda + w = (1+s)z_\lambda + v$, one can easily verify that

$$f(s) = -\big(\frac{s^2}{2} + s + 1\big)z_\lambda\cdot\nabla W(t,z_\lambda) + (1+s)y_\lambda\cdot\nabla W(t,z_\lambda) + W(t,z_\lambda) - W(t,y_\lambda).$$

It is then clear that if $z_\lambda\cdot y_\lambda \leqslant 0$, then (W_6) and (W_8) implies $f(s) < 0$. Suppose that $z_\lambda\cdot y_\lambda > 0$, then in view of (16), (W_{10}) implies $|z_\lambda| = |y_\lambda|$ and by (W_9) we

have $W(t,z_\lambda)=W(t,y_\lambda)$ and $y_\lambda\cdot\nabla W(t,z_\lambda)<z_\lambda\cdot\nabla W(t,z_\lambda)$, whenever $w\neq 0$. This implies that $f(s)<-\frac{s^2}{2}z_\lambda\cdot\nabla W(t,z_\lambda)\leqslant 0$. Hence $f(r)<0$ for every $r\geqslant -1$. It then follows from (15) that $\Phi_\lambda(z_\lambda+w)<\Phi_\lambda(z_\lambda)$. □

Lemma 17. *The sequence $(z_k^n)_n$ obtained in Corollary 15 above is bounded.*

Proof. We assume by contradiction that (z_k^n) is unbounded. Then, up to a subsequence, we may suppose that $\|z_k^n\|\to\infty$ as $n\to\infty$. Let $w_k^n=z_k^n/\|z_k^n\|$, then since (Qw_k^n) is bounded we have either

(i) $(Qw_k^n)_n$ is vanishing, i.e.

$$\lim_{n\to\infty}\sup_{a\in\mathbb{R}}\int_{a-1}^{a+1}|Qw_k^n|^2dt=0,$$

or

(ii) $(Qw_k^n)_n$ is nonvanishing, i.e. there are numbers $r,\delta>0$ and a sequence $(a_n)\subset\mathbb{R}$ such that

$$\liminf_{n\to\infty}\int_{a_n-r}^{a_n+r}|Qw_k^n|^2dt\geqslant\delta.$$

Following an approach by Jeanjean [15], we will find a contradiction by showing that neither (i) nor (ii) does not actually hold.

Assume that (Qw_k^n) is vanishing. Then by Lemma 11 we have $Qw_k^n\to 0$ as $n\to\infty$ in $L^p(\mathbb{R},\mathbb{R}^{2N})$. We then deduce from (5) that for every $R\geqslant 0$, $\int_{\mathbb{R}}W(t,RQw_k^n)dt\to 0$ as $n\to\infty$. Since $\Phi_\lambda(z_k^n)\geqslant 0$, we have $\|Qw_k^n\|\geqslant\|Pw_k^n\|$ and then $\|Qw_k^n\|^2\geqslant\frac{1}{2}$. By using Lemma 16, we have

$$c_k(\lambda_n)=\Phi_{\lambda_n}(z_k^n)\geqslant\Phi_{\lambda_n}(RQw_k^n)\geqslant\frac{R^2}{4}-\lambda_n\int_{\mathbb{R}}W(t,RQw_k^n)dt.$$

Thus by setting $\widetilde{c_k}:=\sup_{u\in B_k}\Phi(u)$, we deduce that

$$\widetilde{c_k}\geqslant\frac{R^2}{4}-\lambda_n\int_{\mathbb{R}}W(t,RQw_k^n)dt\ \to R^2/4\quad\text{as } n\to\infty.$$

We obtain a contradiction by taking R big enough.

Assume now that (Qw_k^n) is not vanishing. Then, up to a translation and a subsequence, we have

$$\liminf_{n\to\infty}\int_{-r-\frac{1}{2}}^{r+\frac{1}{2}}|Qw_k^n|^2dt\geqslant\frac{\delta}{2}. \tag{17}$$

Setting $w_k^n\rightharpoonup w_k$ as $n\to\infty$, then (17) implies, since $Qw_k^n\to Qw_k$ in $L^2_{loc}(\mathbb{R},\mathbb{R}^{2N})$, that $Qw_k\neq 0$. And this implies that $|z_k^n|\to\infty$ as $n\to\infty$. It then follows from (W_7) and Fatou's lemma that

$$\liminf_{n\to\infty}\int_{\mathbb{R}}\frac{W(t,z_k^n)}{\|z_k^n\|^2}dt=\infty.$$

Hence,

$$0\leqslant\frac{\Phi_{\lambda_n}(z_k^n)}{\|z_k^n\|^2}=\frac{1}{2}\big(\|Qw_k^n\|^2-\lambda_n\|Pw_k^n\|^2\big)-\lambda_n\int_{\mathbb{R}}\frac{W(t,z_k^n)}{\|z_k^n\|^2}dt\to-\infty\quad\text{as } n\to\infty.$$

A contradiction again.
Consequently, the sequence $(z_k^n)_n$ is bounded. □

We can now prove Theorem 3.

Proof of Theorem 3. Consider the sequence (z_k^n) above. One can easily verify that

$$\Phi(z_k^n) = \Phi_{\lambda_n}(z_k^n) + \frac{1}{2}(\lambda_n - 1)\|Pz_k^n\|^2 + (\lambda_n - 1)\int_{\mathbb{R}} W(t, z_k^n)dt,$$

and

$$\big\langle \Phi'(z_k^n) - \Phi'_{\lambda_n}(z_k^n), v\big\rangle = (\lambda_n - 1)\Big[(Pz_k^n, v) + \int_{\mathbb{R}} v \cdot W(t, z_k^n)dt\Big].$$

Note that the sequence $(c_k(\lambda_n))_n$ is nondecreasing and bounded from above. Then, there is $c_k \geqslant c_k(1) \geqslant \widetilde{b_k}$ such that $c_k(\lambda_n) \to c_k$ as $n \to \infty$ $\big($where $\widetilde{b_k}$ is defined in (13)$\big)$. It follows from the above relations that (z_k^n) is a $(PS)_{c_k}$ sequence for Φ. By repeating the argument in the proof of Theorem 1, we obtain the existence of $z_k \in X$ such that $\Phi'(z_k) = 0$ and $\Phi(z_k) \geqslant \widetilde{b_k}$. Since $\widetilde{b_k} \to \infty$ as $k \to \infty$, the proof of Theorem 3 is completed. $\square$

4. Existence of a ground state homoclinic solution

4.1. **Generalized Nehari manifold.** Let X be a Hilbert space with norm $\|\cdot\|$, and an orthogonal decomposition $X = X^+ \oplus X^-$. We denote by S^+ the unit sphere in X^+; that is,

$$S^+ := \big\{u \in X^+ \,\big|\, \|u\| = 1\big\}.$$

For $u = u^+ + u^- \in X$, where $u^\pm \in X^\pm$, we define

$$X(u) := \mathbb{R}u \oplus X^- \equiv \mathbb{R}u^+ \oplus X^- \text{ and } \widehat{X}(u) := \mathbb{R}^+u \oplus X^- \equiv \mathbb{R}^+u^+ \oplus X^-. \quad (18)$$

Let Φ be a $\mathcal{C}^1-$functional defined on X by

$$\Phi(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - P(u).$$

We consider the following situation:

- (H_1) $P(0) = 0$, $\frac{1}{2}\big\langle P'(u), u\big\rangle > P(u) > 0$ for all $u \neq 0$ and P is weakly lower semicontinuous.
- (H_2) For each $w \in X\backslash X^-$, there exists a unique nontrivial critical point of $\widehat{m}(w)$ of $\Phi|_{\widehat{X}(w)}$, which is the unique global maximum of $\Phi|_{\widehat{X}(w)}$.
- (H_3) There exists $\delta > 0$ such that $\|\widehat{m}(w)^+\| \geqslant \delta$ for all $w \in X\backslash X^-$, and for each compact subset $\mathcal{K} \subset X\backslash X^-$, there exists a constant $C_{\mathcal{K}}$ such that $\|\widehat{m}(w)\| \leqslant C_{\mathcal{K}}$.

The following set was introduced by Pankov [17]:

$$\mathcal{M} := \big\{u \in X\backslash X^- : \big\langle \Phi'(u), u\big\rangle = 0 \text{ and } \big\langle \Phi'(u), v\big\rangle = 0\ \forall v \in X^-\big\}.$$

It is called the generalized Nehari manifold.

Remark 18. By (H_1), $\mathcal{M}$ contains all nontrivial critical points of Φ and by (H_2), $\widehat{X}(w) \cap \mathcal{M} = \{\widehat{m}(w)\}$ whenever $w \in X\backslash X^-$.

We also consider the mappings:

$$\widehat{m} : X\backslash X^- \to \mathcal{M},\ w \mapsto \widehat{m}(w) \text{ and } m := \widehat{m}|_{S^+}.$$

$$\widehat{\Psi} : X^+\backslash\{0\} \to \mathbb{R},\ \widehat{\Psi}(w) := \Phi(\widehat{m}(w)) \text{ and } \Psi := \widehat{\Psi}|_{S^+}.$$

The following result is due to A. Szulkin and T. Weth $\big($[23], Corollary 33$\big)$.

Theorem 19. *If (H_1), (H_2) and (H_3) are satisfied, then*

(a) $\Psi \in \mathcal{C}^1(S^+,\mathbb{R})$ *and*

$$\langle \Psi'(w), z\rangle = \|m(w)^+\|\langle \Phi'(m(w)), z\rangle \text{ for all } z \in T_w(S),$$

where $T_w(S)$ is the tangent space of S at w.

(b) *If (w_n) is a Palais-Smale sequence for Ψ, then $(m(w_n))$ is a Palais-Smale sequence for Φ. If $(u_n) \subset \mathcal{M}$ is a bounded Palais-Smale sequence for Φ, then $(m^{-1}(w_n))$ is a Palais-Smale sequence for Ψ.*

(c) *w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of Φ. Moreover, the corresponding critical values coincide and $inf_{S^+}\Psi = inf_{\mathcal{M}}\Phi$.*

4.2. **Existence of a ground state.** Throughout this subsection, we assume that (A_0), $(W_1) - (W_3)$ and $(W_6) - (W_{10})$ are satisfied.
Here P is given by

$$P(u) := \int_{\mathbb{R}} W(t,u)dt.$$

Lemma 20. *Condition (H_1) is satisfied.*

Proof. Clearly we have $P(0) = 0$ and $\frac{1}{2}\langle P'(u), u\rangle \geqslant P(u) > 0$ for any $u \neq 0$. Let $(u_n) \subset X$ and $C \in \mathbb{R}$ such that $u_n \rightharpoonup u$ and $N(u_n) \leqslant C$. Since the embedding of X in $L^2_{loc}(\mathbb{R},\mathbb{R}^{2N})$ is compact, we have $u_n \to u$ in $L^2_{loc}(\mathbb{R},\mathbb{R}^{2N})$ and up to a subsequence $u_n \to u$ a.e.. It then follows from Fatou's lemma that $P(u) \leqslant C$. Hence P is weakly lower semicontinuous. □

Lemma 21. *Condition (H_2) is satisfied.*

Proof. Let $w \in X\backslash X^-$. Then there exists R large enough such that $\Phi \leqslant 0$ on $\widehat{X}(w)\backslash B_R$, where $B_R := \{u \in X \mid \|u\| \leqslant R\}$. In fact, if this is not true then there exists a sequence $(u_n) \subset \widehat{X}(w)$ such that $\|u_n\| \to \infty$ and $\Phi(u_n) > 0$. Up to a subsequence we have $v_n = u_n/\|u_n\| \rightharpoonup v$ in X. By (2) we have

$$0 < \frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_\Omega \frac{F(x,\|u_n\|v_n)}{\left|v_n\|u_n\|\right|^2}|v_n|^2.$$

If $v \neq 0$, we deduce by using Fatou's Lemma and (W_7) that $0 \leqslant -\infty$; a contradiction. Consequently $v = 0$. Since $\widehat{X}(w) = \widehat{X}(w^+/\|w^+\|)$, we may assume that $w \in S^+$. Now, since $P(u_n) \geqslant 0$ and $1 = \|v_n^+\|^2 + \|v_n^-\|^2$, then necessarily $v_n^+ = s_n w \nrightarrow 0$. Hence there is $r > 0$ such that $\|v_n^+\| = \|s_n w\| > r$ $\forall n$. So $\|v_n^+\| = s_n$ is bounded and bounded away from 0. But then, up to a subsequence, $v_n^+ \to sw$, $s > 0$, which contradicts the fact that $v_n \rightharpoonup 0$.
By (W_3), $\Phi(sw) = \frac{1}{2}s^2 + \circ(s^2)$ as $s \to 0$. Hence $0 < \sup_{\widehat{X}(w)} \Phi < \infty$. Since Φ is weakly upper semicontinuous on $\widehat{X}(w)$ and $\Phi \leqslant 0$ on $\widehat{X}(w) \cap X^-$, the supremum is attained at some point u_0 such that $u_0^+ \neq 0$. So u_0 is a nontrivial critical point of $\Phi|_{\widehat{X}(w)}$ and hence $u_0 \in \mathcal{M}$.

We will now show that if $u \in \mathcal{M}$, then u is the unique global maximum of $\Phi|_{\hat{X}(u)}$. Let $u \in \mathcal{M}$ and $w = u + w \in \hat{X}(u)$ with $w \neq 0$. By the definition of $\hat{X}(u)$, we have $u + w = (1+s)u + v$, with $s \geqslant -1$ and $v \in X^-$. Using the argument in the proof of Lemma 16, we see that $\Phi(u + w) < \Phi(u)$. □

Lemma 22. *Condition (H_3) is satisfied.*

Proof. (W_3) and (H_1) imply that

$$\forall \varepsilon > 0,\, \exists \alpha > 0,\, |u| < \alpha \implies P(u) < \frac{1}{2}\big\langle P'(u), u\big\rangle \leqslant \frac{\varepsilon}{2}\|u\|^2.$$

Hence, there exist $\rho, \eta > 0$ such that $\Phi(w) \geqslant \eta$ for every $w \in \{u \in X^+ \mid \|u\| = \rho\}$. By (H_2), we have $\Phi(\hat{m}(w)) \geqslant \eta$ for every $w \in X\backslash X^-$. Since $P \geqslant 0$, we deduce from (2) that $\|\hat{m}(w)^+\| \geqslant \sqrt{2\eta}\ \forall w \in X\backslash X^-$.

Now let $\mathcal{K}$ be a compact subset of $X\backslash X^-$. We claim that there exists a constant $C_{\mathcal{K}}$ such that $\|\widehat{m}(w)\| \leqslant C_{\mathcal{K}}$, $\forall w \in \mathcal{K}$. In fact, if the claim is not true, then we can find a subsequence $(w_n) \subset \mathcal{K}$ such that $\|\widehat{m}(w_n)\| \to \infty$ as $n \to \infty$. Since $\widehat{m}(w) = \widehat{m}(w^+/\|w^+\|)\ \forall w \in X\backslash X^-$, we may assume that $\mathcal{K} \subset S^+$. By using the fact that $\widehat{m}(w_n) \in \mathcal{M}$, one can verify easily that $\Phi(\widehat{m}(w_n)) > 0$. Define $y_n = \widehat{m}(w_n)/\|\widehat{m}(w_n)\|$. Then we have

$$0 \leqslant \frac{\Phi(\widehat{m}(w_n))}{\|\widehat{m}(w_n)\|^2} = \frac{1}{2}\Big(\frac{\|\widehat{m}(w_n)^+\|^2}{\|\widehat{m}(w_n)\|^2} - \frac{\|\widehat{m}(w_n)^-\|^2}{\|\widehat{m}(w_n)\|^2}\Big) - \int_{\mathbb{R}} \frac{W(t, y_n\|\widehat{m}(w_n)\|)}{|y_n\|\widehat{m}(w_n)\||^2}|y_n|^2 dt.$$

Since $y_n \in \hat{X}(u)$, then $y_n = s_n w_n + v_n$, with $s_n \geqslant 0$ and $v_n \in X^-$. It follows that

$$0 \leqslant \frac{\Phi(\widehat{m}(w_n))}{\|\widehat{m}(w_n)\|^2} = \frac{1}{2}\big(\lambda_n^2 - \|v_n\|^2\big) - \int_{\mathbb{R}} \frac{W(t, y_n\|\widehat{m}(w_n)\|)}{|y_n\|\widehat{m}(w_n)\||^2}|y_n|^2 dt. \tag{19}$$

Since $W \geqslant 0$, we deduce that $s_n^2 \geqslant \|v_n\|^2$ and then $\frac{1}{\sqrt{2}} \leqslant s_n \leqslant 1$. Up to a subsequence, $s_n \to s > 0$ and $w_n \to w \in S^+$. Hence $y_n \rightharpoonup y \neq 0$. If we take the limit $n \to \infty$ in (19), we obtain by using (W_7) and Fatou's lemma the contradiction $0 \leqslant -\infty$. ☐

Lemma 23. *There exists* $\alpha > 0$ *such that*

$$c = \inf_{\mathcal{M}} \Phi \geqslant \inf_{S_\alpha} \Phi > 0,$$

where $S_\alpha := \{u \in X^+ \,;\, \|u\| = \alpha\}$.

Proof. We remark that we can choose ε in (5) in such a way that

$$\Phi(u) \geqslant \frac{1}{4}\|u\|^2 - C\|u\|^p, \quad \forall u \in X^+.$$

It suffices to take α sufficiently small. ☐

Proof of Theorem 4. By the preceding lemmas we know that (H_1), (H_2) and (H_3) are satisfied. By Theorem 19-(a), $\Psi \in \mathcal{C}^1(S^+, \mathbb{R})$. The Ekeland variational principle [27] then gives the existence of a sequence $(w_n) \subset S^+$ such that $\Psi(w_n) \to \inf_{S^+} \Psi$. By Theorem 19-$(b)$, $\big(u_n := \hat{m}(w_n)\big)$ is a Palais-Smale sequence for Φ on $\mathcal{M}$. By using the argument in the proof of Lemma 17, we show that (u_n) is bounded. Up to a subsequence, $u_n \rightharpoonup u$ in X. We claim that $u_n \nrightarrow 0$ in $L^p(\mathbb{R}, \mathbb{R}^{2N})$. In fact, if this is not true, then we deduce from (5) that $\int_{\mathbb{R}} u_n^+ \cdot W(t, u_n)dt \to 0$ as $n \to \infty$. It follows that

$$\|u_n^+\|^2 = \big\langle \Phi'(u_n), u_n^+\big\rangle - \int_{\mathbb{R}} u_n^+ \cdot W(t, u)dt \to 0 \quad \text{as } n \to \infty.$$

But since $\Phi(u_n) \leqslant \frac{1}{2}\|u_n^+\|^2$, we deduce that $\liminf_{n\to\infty} \Phi(u_n) = 0$, which contradicts Lemma 23. Hence $u_n \nrightarrow 0$ in $L^p(\mathbb{R}, \mathbb{R}^{2N})$.

By Lemma 11, there exist $\delta > 0$ and $(a_n) \subset \mathbb{R}_+$ such that

$$\int_{-a_n}^{a_n} |u_n|^2 dt \geqslant \delta.$$

Up to translation and a subsequence, we deduce that $u \neq 0$. Now since Φ' is weakly sequentially continuous, we obtain $\Phi'(u) = 0$, that is, u is a non trivial weak solution of (HS). On the other hand, we may assume that $u_n \to u$ a.e., which together with (W_8) and Fatou's lemma imply, since

$$\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n \rangle \;=\; \int_{\mathbb{R}} \Big(\frac{1}{2} u_n \cdot \nabla W(t, u_n) - W(t, u_n)\Big) dt,$$

that

$$\inf_{\mathcal{M}} \Phi \geqslant \int_{\mathbb{R}} \big(\frac{1}{2} u \cdot \nabla W(t, u) - W(t, u)\big) dt = \Phi(u) - \frac{1}{2}\langle \Phi'(u), u \rangle.$$

This implies $\Phi(u) \leqslant \inf_{\mathcal{M}} \Phi$. Since $u \in \mathcal{M}$, the reverse inequality also holds and therefore

$$\Phi(u) = \inf_{\mathcal{M}} \Phi.$$

□

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